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UNCONDITIONAL BASES AND UNCONDITIONAL FINITE-DIMENSIONAL DECOMPOSITIONS IN BANACH SPACES

BY

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ABSTRACT

Let X be a Banach space with an unconditional finite-dimensional Schauder decomposition (E_n). We consider the general problem of characterizing conditions under which one can construct an unconditional basis for X by forming an unconditional basis for each E_n. For example, we show that if $\sup_n \dim E_n < \infty$ and X has Gordon–Lewis local unconditional structure then X has an unconditional basis of this type. We also give an example of a non-Hilbertian space X with the property that whenever Y is a closed subspace of X with a UFDD (E_n) such that $\sup_n \dim E_n < \infty$ then Y has an unconditional basis, showing that a recent result of Komorowski and Tomczak-Jaegermann cannot be improved.

1. Introduction

Let X be a separable Banach space with an unconditional finite-dimensional Schauder decomposition (UFDD) (E_n). It is well-known that even if for some constant K each E_n has a K-unconditional basis it is still possible that X may fail to have an unconditional basis. The first example of this phenomenon was given in [10] where a twisted sum of two Hilbert spaces Z_2 is constructed in such a way that it has a UFDD into a two-dimensional spaces (or a 2-UFDD) E_n but Z_2 has no unconditional basis. Later, Johnson, Lindenstrauss and Schechtman

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[6] showed that this same example fails even to have local unconditional structure (l.u.st.).

Recently, Komorowski and Tomczak-Jaegermann [13] proved the remarkable result that if X has an unconditional basis and is not hereditarily Hilbertian then it has a subspace Y with a 2-UFDD and failing local unconditional structure. This is an important step in the resolution of the conjecture that a Banach space all of whose subspaces have local unconditional structure must be Hilbertian.

Motivated by these results, we investigate here the construction of unconditional bases or unconditional basic sequences in spaces with a UFDD. For convenience let us refer to a UFDD (E_n) as **uniform** if $\sup_n \dim E_n < \infty$ and as an N-UFDD if $\dim E_n = N$ for all n.

Suppose X has an unconditional basis and the property that whenever (E_n) is a UFDD for X and, for each n, $(f_{ni})_{i=1}^{\dim E_n}$ is a basis of E_n with unconditional basis constant (ubc) bounded by some constant K, then $(f_{ni})_{n,i}$ forms an unconditional basis of X. In Section 2 we prove that this property characterizes the spaces ℓ_1, ℓ_2 and c_0. A similar property for any UFDD of a closed subspace characterizes ℓ_2.

Now suppose X is a Banach space with a uniform UFDD (E_n). Under these hypotheses we show that (Gordon–Lewis) l.u.st. is equivalent to the existence of an unconditional basis for X of the form $(f_{ni})_{n,i}$ where each $(f_{ni})_{i=1}^{\dim E_n}$ is an unconditional basis for E_n. This provides us with a simple criterion to check whether a given space with a uniform UFDD has l.u.st.: compare the earlier criteria used by Ketonen [11], Borzyszkowski [3] and Komorowski [12]. Using this criterion we establish a general result on the failure of l.u.st. in twisted sums.

Finally in Section 4, we give an example to complement the work of Komorowski and Tomczak-Jaegermann [13]. We show that there is an Orlicz sequence space $\ell_F \neq \ell_2$ with the property that whenever (E_n) is a uniform UFDD for a closed subspace X_0 then one can choose an unconditional basis $(f_{ni})_{i=1}^{\dim E_n}$ of each E_n so that the family $(f_{ni})_{n,i}$ is an unconditional basis of X_0. Of course the space ℓ_F is hereditarily Hilbertian; this example shows that the result of [13] is in a sense best possible.

2. Preliminary results

Let us say that a UFDD (E_n) is **absolute** if there is a constant C so that if $y_n, x_n \in E_n$ are finitely nonzero and satisfy $\|y_n\| \leq \|x_n\|$ for all n then

$\|\sum_{n=1}^{\infty} y_n\| \leq C\|\sum_{n=1}^{\infty} x_n\|$. We remark that in [2] it is shown that every FDD of a reflexive subspace of a space with a shrinking UFDD, for which every blocking is absolute, can be blocked to be a UFDD. In [1] there is a more technical result which extends this and also gives some conditions under which one can construct an unconditional basis for the subspace. The following Proposition is trivial:

PROPOSITION 2.1: *Suppose (E_n) is an absolute UFDD of a Banach space X and that $(f_{ni})_{i=1}^{\dim E_n}$ is an unconditional basis of E_n so that $\sup_n$ ubc $(f_{ni}) < \infty$. Then $(f_{ni})_{n,i}$ is an unconditional basis of X.*

PROPOSITION 2.2: *Let (E_n) be a UFDD of a Banach space X such that there is an unconditional basis $(g_{ni})_{i=1}^{\dim E_n}$ for each E_n with $\sup_n$ ubc $(g_{ni})_{i=1}^{\dim E_n} < \infty$. Suppose further that whenever we pick an unconditional basis $(f_{ni})_{i=1}^{\dim E_n}$ of E_n in such a way that $\sup_n$ ubc $(f_{ni})_{i=1}^{\dim E_n} < \infty$, then $(f_{ni})_{n,i}$ forms an unconditional basis of X. Then (E_n) is an absolute UFDD.*

Proof: Suppose (E_n) is not an absolute UFDD. Then by a gliding hump argument we can find two normalized sequences $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ so that $x_n, y_n \in F_n$ and so that it is not true that for some constant C and any finitely nonzero sequence of scalars $(\alpha_n)_{n=1}^{\infty}$ we have

$$\|\sum_{n=1}^{\infty} \alpha_n y_n\| \leq C\|\sum_{n=1}^{\infty} \alpha_n x_n\|.$$

Let $F_n = [x_n, y_n]$ so that $\dim F_n \leq 2$. Then there is a projection $P_n\colon E_n \to F_n$ with $\|P_n\| \leq \sqrt{2}$; let G_n be the complementary space. Then if G_n is nontrivial, since it has codimension in E_n of at most two, it is at least 9-isomorphic to $[g_{ni}]_{i=1}^{\dim G_n}$ (see Zippin [20]). Hence if we select any unconditional basis $(f_{ni})_{i=1}^{\dim F_n}$ of each F_n with sup ubc $(f_{ni})_{i=1}^{\dim F_n} < \infty$ we can extend it to an unconditional basis of E_n. It then follows that $(f_{ni})_{n,i}$ is an unconditional basic sequence.

Since each F_n has dimension at most two, we can introduce a Euclidean norm $\| \ \|_{F_n}$ so that $\|x\| \leq \|x\|_{F_n} \leq \sqrt{2}\|x\|$ for $x \in F_n$. Let $(\ , \ )_{F_n}$ be the associated inner-product. We will show that if $\xi_n, \eta_n \in F_n$ with $\|\xi_n\|_{F_n} = \|\eta_n\|_{F_n} = 1$ then there is a constant C so that for any finitely nonzero sequence (α_n) we have

$$\|\sum_{n=1}^{\infty} \alpha_n \eta_n\| \leq C\|\sum_{n=1}^{\infty} \alpha_n \xi_n\|. \tag{1}$$

It plainly suffices to consider the case where $(\xi_n, \eta_n)_{F_n}$ is real and non-negative. Then $\|\xi_n + \eta_n\|_{F_n} = b_n \geq \sqrt{2}$. Let $\zeta_n = b_n^{-1}(\xi_n + \eta_n)$. Then we can extend (ζ_n)

to an orthonormal basis of F_n and by the above remarks there is a constant C_1 so that if $(\alpha_n)_{n=1}^\infty$ is finitely nonzero then

$$\|\sum_{n=1}^\infty \alpha_n(\xi_n,\zeta_n)_{F_n}\zeta_n\| \le C_1\|\sum_{n=1}^\infty \alpha_n\xi_n\|.$$

By a similar argument, there is a constant C_2 so that

$$\|\sum_{n=1}^\infty \alpha_n(\xi_n,\zeta_n)_{F_n}(\zeta_n,\eta_n)_{F_n}\eta_n\| \le C_2\|\sum_{n=1}^\infty \alpha_n\xi_n\|.$$

Now $(\xi_n,\zeta_n)_{F_n}(\zeta_n,\eta_n)_{F_n} \ge \frac{1}{2}$ and this establishes the desired inequality (1), which clearly leads to a contradiction if we set $\xi_n = x_n/\|x_n\|_{F_n}$ and $\eta_n = y_n/\|y_n\|_{F_n}$. ∎

We shall say that an unconditional basis $(e_n)_{n=1}^\infty$ for a Banach space X has the **shift property** (SP) if whenever (x_n) is a normalized block basic sequence there is a constant C so that for any finitely nonzero sequence (α_n) we have

$$C^{-1}\|\sum_{n=1}^\infty \alpha_n x_n\| \le \|\sum_{n=1}^\infty \alpha_n x_{n+1}\| \le C\|\sum_{n=1}^\infty \alpha_n x_n\|. \tag{2}$$

It is easy to see that if X has (SP) then there is a uniform constant C so that (2) holds for all normalized block basic sequences. We also remark that, although our formulation is mildly different, essentially the same concept was introduced for sequence spaces in [9]. Precisely, the unconditional basis (e_n) has (SP) if and only if the corresponding sequence space has both the left-shift (LSP) and right-shift (RSP) properties. No example of a sequence space with just one shift property, say (LSP), and not the other is known.

Proposition 2.3: *The following properties are equivalent:*

1. $(e_n)_{n=1}^\infty$ *has property (SP).*
2. *For every blocking* $E_n = [e_i]_{i=r_{n-1}+1}^{r_n}$ *of* (e_n) *and every unconditional basis* $(f_k)_{k=r_{n-1}+1}^{r_n}$ *of* E_n *such that* $\sup_n \mathrm{ubc}\,(f_k)_{k=r_{n-1}+1}^{r_n} < \infty$ *the sequence* $(f_k)_{k=1}^\infty$ *forms an unconditional basis of* X.
3. *For every blocking* E_n *of* (e_n) *and every sequence* (F_n) *of 2-dimensional spaces so that* $F_n \subset E_n$ *and every unconditional basis* (f_{2n-1},f_{2n}) *of* F_n *with* $\sup_n \mathrm{ubc}\,(f_{2n-1},f_{2n}) < \infty$ *the sequence* $(f_n)_{n=1}^\infty$ *is an unconditional basic sequence.*

Proof: Clearly (1) implies easily that every blocking (E_n) is an absolute UFDD, and so implies (2) by Proposition 2.1. (2) also implies by Proposition 2.2 that every blocking is absolute and so also implies (3) by Proposition 2.1.

Finally suppose we have (3). Suppose (x_n) is a normalized block basic sequence. It follows from Proposition 2.2 and (3) that the UFDD $F_n = [x_{2n-1}, x_{2n}]$ is absolute. Hence, for a suitable constant C_0, and for any finitely nonzero sequence (α_n) we have:

$$C_0^{-1} \| \sum_{n=1}^{\infty} \alpha_{2n-1} x_{2n} \| \leq \| \sum_{n=1}^{\infty} \alpha_{2n-1} x_{2n-1} \| \leq C_0 \| \sum_{n=1}^{\infty} \alpha_{2n-1} x_{2n} \|.$$

In a similar fashion, considering $G_n = [x_{2n}, x_{2n+1}]$ we have a constant C_1 so that:

$$C_1^{-1} \| \sum_{n=1}^{\infty} \alpha_{2n} x_{2n+1} \| \leq \| \sum_{n=1}^{\infty} \alpha_{2n} x_{2n} \| \leq C_1 \| \sum_{n=1}^{\infty} \alpha_{2n} x_{2n+1} \|.$$

Combining this with the fact that (x_n) is an unconditional basic sequence shows that (e_n) has the shift property (SP). ∎

THEOREM 2.4: *Let X be a Banach space with an unconditional basis. Then the following are equivalent:*

(1) *X is isomorphic to one of the spaces ℓ_1, ℓ_2 or c_0.*
(2) *Whenever (E_n) is a UFDD for X and $(f_{ni})_{i=1}^{\dim E_n}$ is an unconditional basis for each E_n with $\sup_n$ ubc $(f_{ni})_{i=1}^{\dim E_n} < \infty$ then $(f_{ni})_{n,i}$ is an unconditional basis for X.*

Proof: (1) $\Rightarrow$ (2). This is obtained by putting together some folklore results. It follows easily from the parallelogram law that if (E_n) is a UFDD for ℓ_2 then there is a constant C so that if $(x_n)_{n=1}^{\infty}$ is a finitely nonzero sequence with $x_n \in E_n$ then

$$C^{-1} (\sum_{n=1}^{\infty} \|x_n\|^2)^{1/2} \leq \| \sum_{n=1}^{\infty} x_n \| \leq C (\sum_{n=1}^{\infty} \|x_n\|^2)^{1/2}.$$

If (E_n) is a UFDD for ℓ_1 one obtains the similar inequality

$$C^{-1} \sum_{n=1}^{\infty} \|x_n\| \leq \| \sum_{n=1}^{\infty} x_n \| \leq C \sum_{n=1}^{\infty} \|x_n\|,$$

from the classical argument of Lindenstrauss–Pełczyński [14] that the unconditional basis of ℓ_1 is unique. In the case of c_0 one obtains

$$C^{-1} \max_{1 \leq n < \infty} \|x_n\| \leq \| \sum_{n=1}^{\infty} x_n \| \leq C \max_{1 \leq n < \infty} \|x_n\|.$$

In all cases the UFDD is absolute and Proposition 2.1 gives the conclusion.

(2) $\Rightarrow$ (1). It follows from Proposition 2.3 that every permutation of every unconditional basis has the shift property. Thus any unconditional basis (e_n) is a symmetric basis with the (SP) and so by Proposition 2.3 of [9] X is isomorphic to one of the spaces ℓ_p for $1 \leq p < \infty$ or to c_0. Since every unconditional basis is symmetric this shows that X is isomorphic to one of the three spaces ℓ_1, ℓ_2 or c_0 (cf. [18]). ∎

COROLLARY 2.5: *Let X be a Banach space with an unconditional basis. Then the following are equivalent:*

(1) *X is isomorphic to ℓ_2.*
(2) *Whenever (E_n) is a UFDD for a closed subspace Y of X and $(f_{ni})_{i=1}^{\dim E_n}$ is an unconditional basis for each E_n with $\sup_n$ ubc $(f_{ni})_{i=1}^{\dim E_n} < \infty$ then $(f_{ni})_{n,i}$ is an unconditional basis for Y.*
(3) *Whenever (E_n) is a 2-UFDD for a closed subspace Y of X and $(f_{ni})_{i=1,2}$ is an unconditional basis for each E_n with $\sup_n$ ubc $(f_{ni})_{i=1,2} < \infty$ then $(f_{ni})_{n,i}$ is an unconditional basis for Y.*

Proof: Clearly (1) implies (2) and (2) implies (3). For (3) $\Rightarrow$ (1) we use Proposition 2.3 to deduce that every unconditional basic sequence has the shift-property and hence as in Theorem 2.4, X is isomorphic to ℓ_1, ℓ_2 or c_0. Since this property passes to every closed subspace we obtain X isomorphic to ℓ_2. ∎

Our final result of the section is that if one can choose an unconditional basis from a uniform UFDD then it is essentially unique.

PROPOSITION 2.6: *Suppose X is a Banach space with a uniform UFDD (E_n). Suppose $(f_{ni})_{i=1}^{\dim E_n}$ and $(g_{ni})_{i=1}^{\dim E_n}$ are normalized unconditional bases for each E_n, so that the whole collections $(f_{ni}), (g_{ni})$ are unconditional bases of X. Then (f_{ni}) and (g_{ni}) are permutatively equivalent.*

Proof: Let $d_n = \dim E_n$. Let $g_{ni} = \sum_{j=1}^{d_n} a_{ij}^n f_{nj}$. It is easy to see that $\inf_n |\det (a_{ij}^n)| > 0$ and so for some $c > 0$ and each n, there is a permutation σ_n of $\{1, 2, \ldots, d_n\}$ so that $|a_{i,\sigma_n(i)}| > c$.

Now by Krivine's theorem, for any finitely nonzero (α_{ni})

$$\|\sum_{n,j} (\sum_i |\alpha_{ni}|^2 |a_{ij}^n|^2)^{1/2} f_{nj}\| \leq C \|\sum_{n,i} \alpha_{ni} g_{ni}\|$$

where C is a suitable constant. Therefore,

$$\|\sum_{n,i}\alpha_{ni}f_{n,\sigma_n(i)}\| \le Cc^{-1}\|\sum_{n,i}\alpha_{ni}g_{ni}\|.$$

Thus the basis (g_{ni}) dominates the basis $(f_{n,\sigma_n(i)})$. By the same argument, there exist permutations τ_n of $\{1,2,\dots,d_n\}$ so that (f_{ni}) dominates $(g_{n,\tau_n(i)})$. Thus f_{ni} dominates $f_{n,\pi_n(i)}$, where $\pi_n = \sigma_n\tau_n$. Iterating $N!$ times where $N = \sup d_n$, since $\pi_n^{N!}$ is always the identity permutation, this implies that (f_{ni}) and $(f_{n,\pi(i)})$ are actually equivalent and so (f_{ni}) and $(g_{n\sigma_n(i)})$ are equivalent. ∎

3. Spaces with local unconditional structure and a UFDD

Let Y be a space with an unconditional basis (e_n). We shall say that a sequence of finite-dimensional subspaces (E_n) forms a **complemented block UFDD** if there is an increasing sequence of integers $(p_n)_{n=0}^{\infty}$ with $p_0 = 0$ so that $E_n \subset F_n = [e_j]_{j=p_{n-1}+1}^{p_n}$ and a projection P on Y so that $P(F_n) = E_n$. If further $\sup\dim F_n < \infty$ then (E_n) is a **uniform complemented block UFDD**.

LEMMA 3.1: *If (E_n) is a uniform complemented block UFDD then one can choose an unconditional basis $(f_{nj})_{j=1}^{\dim E_n}$ in each E_n so that $(f_{nj})_{n,j}$ is an unconditional basis for the closed linear span $X = \sum_{n=1}^{\infty} E_n$. Furthermore (f_{nj}) is equivalent in a suitable order to a subsequence of (e_n).*

Proof: We shall prove the first statement by induction on $M = \sup_n \dim E_n$; it is clearly true if $M = 1$. Assume the statement true whenever $\sup_n \dim E_n < M$ and suppose $\sup_n \dim E_n = M$. We first show that it is possible to pick normalized vectors $f_{n1} \in E_n$ so that there is a projection $Q\colon X \to [f_{n1}]$ with $Q(E_n) \subset E_n$, for each n. To see this note that for each n we have

$$\sum_{k=p_{n-1}+1}^{p_n} \langle Pe_n, e_n^*\rangle = \dim E_n$$

and so there exists $p_{n-1} < k_n \le p_n$ so that $\alpha_n = \langle Pe_{k_n}, e_{k_n}^*\rangle > N^{-1}$ where $N = \sup(p_n - p_{n-1})$. Let $f_{n1} = Pe_{k_n}$ and consider the projection $Q\colon Y \to Y$ defined by $Qy = \sum_{n=1}^{\infty}\alpha_n^{-1}\langle y, e_{k_n}^*\rangle f_{n1}$. It is readily verified that Q is bounded.

Let $G_n = E_n \cap Q^{-1}\{0\}$. Then, after deleting trivial spaces, (G_n) is a uniform complemented block UFDD with $\sup_n \dim G_n \le M - 1$. We therefore can pick an unconditional basis $(f_{nj})_{j=2}^{\dim E_n}$ in each so that $(f_{nj})_{n,j}$ is an unconditional basis of X.

To complete the proof let $H_n = P^{-1}\{0\} \cap F_n$. Then (H_n) is also a uniform complemented block UFDD. It is therefore possible to extend $(f_{nj})_{j=1}^{\dim E_n}$ to an unconditional basis $(f_{nj})_{j=1}^{\dim F_n}$ of (F_n) in such a way that $(f_{nj})_{n,j}$ is an unconditional basis of Y. The final statement follows from Proposition 2.6. ∎

LEMMA 3.2: *Let X be a finite-dimensional Banach space. Suppose $X = E_1 \oplus E_2 \oplus \cdots E_n$ with associated projections $Q_j\colon X \to E_j$ satisfying*

$$\sup_{|\alpha_j|\le 1} \|\sum_{j=1}^{n} \alpha_j Q_j\| = K.$$

Suppose Y is a finite-dimensional Banach lattice and that $A\colon X \to Y$ and $B\colon Y \to X$ are operators so that $BA = I_X$. Then there is a finite-dimensional Banach lattice Z with a band decomposition $Z = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_n$ and operators $A_0\colon X \to Z$, $B_0\colon Z \to X$ with $B_0A_0 = I_X$, $A_0(E_j) \subset Z_j$, $B_0(Z_j) \subset E_j$ and $\|A_0\|\|B_0\| \le 2K^2\|A\|^2\|B\|^2$.

Proof: Consider $Z = Y^n$ with the lattice seminorm

$$\|(y_1, \dots, y_n)\|_Z = \sup_{\|y^*\|\le 1} \sum_{j=1}^{n} \langle |y_j|, |B^*Q_j^*A^*y^*|\rangle.$$

(Strictly speaking Z should be replaced by its Hausdorff quotient.) We define $A_0\colon X \to Z$ by $A_0x = (AQ_1x, \dots, AQ_nx)$ and $B_0\colon Z \to X$ by $B_0(y_1, \dots, y_n) = \sum_{j=1}^{n} Q_jBy_j$. Clearly B_0, A_0 satisfy all the required properties except possibly the norm estimates.

If $x \in X$ and $y^* \in Y^*$ with $\|y^*\| \le 1$ then

$$\sum_{j=1}^{n} \langle |AQ_jx|, |B^*Q_j^*A^*y^*|\rangle \le \langle (\sum_{j=1}^{n} |AQ_jx|^2)^{1/2}, (\sum_{j=1}^{n} |B^*Q_j^*A^*y^*|^2)^{1/2}\rangle.$$

Now, by Khintchine's inequality,

$$\|(\sum_{j=1}^{n} |AQ_jx|^2)^{1/2}\| \le 2^{1/2} \operatorname*{Ave}_{\epsilon_j=\pm 1} \|\sum_{j=1}^{n} \epsilon_j AQ_jx\| \le 2^{1/2}K\|A\|\|x\|.$$

Similarly,

$$\|(\sum_{j=1}^{n} |B^*Q_j^*A^*y^*|^2)^{1/2}\| \le 2^{1/2}K\|A\|\|B\|.$$

It follows that $\|A_0\| \leq 2K^2\|A\|^2\|B\|$.

On the other hand if $x^* \in X^*$ with $\|x^*\| \leq 1$ and if $(y_1, y_2, \ldots, y_n) = z \in Z$ then

$$\begin{aligned} |\langle B_0 z, x^* \rangle| &\leq \sum_{j=1}^{n} |\langle y_j, B^* Q_j^* x^* \rangle| \\ &\leq \sum_{j=1}^{m} \langle |y_j|, |B^* Q_j^* A^* B^* x^*| \rangle \\ &\leq \|B\| \|z\|. \end{aligned}$$

Hence $\|A_0\|\|B_0\| \leq 2K^2\|A\|^2\|B\|^2$. ∎

We now recall that a Banach space X has Gordon–Lewis local unconditional structure (or l.u.st.) [5] if there is a constant C so that whenever E is a finite-dimensional subspace of X there is a finite-dimensional Banach lattice Y and operators $A\colon E \to Y$, $B\colon Y \to X$ with $\|A\|\|B\| \leq C$ and $BA = I_E$. (A stronger form of local unconditional structure is considered in [4].)

The following Proposition is established by Johnson, Lindenstrauss and Schechtman [6], under the additional assumptions that X has nontrivial cotype and is complemented in its bidual.

PROPOSITION 3.3: *Let X be a Banach space with a UFDD (E_n). Suppose X has local unconditional structure. Then there a Banach space Y with an unconditional basis (e_n) so that Y contains X and (E_n) is a complemented block UFDD.*

Proof: We let Q_n be the natural projection of X onto E_n and set $K = \sup_n \|Q_n\|$. Let $X_n = \sum_{j=1}^{n} E_j$. Using l.u.st. and the preceding Lemma, there is a constant C so that for each n we can find a finite-dimensional Banach lattice Z_n with a band decomposition $Z_n = Z_{n1} \oplus \cdots \oplus Z_{nn}$ and operators $A_n\colon X_n \to Z_n$, $B_n\colon Z_n \to X_n$ so that $B_n A_n = I_{X_n}$, $\|A_n\| \leq 1$, $\|B_n\| \leq C$ and $A_n(E_j) \subset Z_{nj}$, $B_n(Z_{nj}) \subset E_j$ for $1 \leq j \leq n$.

Choose $\epsilon_n > 0$ to be a sequence such that $\sum \epsilon_n < (2C)^{-1}$. Then by an argument of Johnson [17] we can find for each n and each $1 \leq j \leq n$ a sublattice Y_{nj} of Z_{nj} with $\dim Y_{nj} \leq d_j$ (independent of n) and a map $L_{nj}\colon A_n(E_j) \to Y_{nj}$ so that $\|L_{nj} z - z\| \leq \epsilon_j \|z\|$.

Now let $Y_n = Y_{n1} \oplus \cdots \oplus Y_{nn}$ and define $\tilde{A}_n\colon X_n \to Y$ by $\tilde{A}_n x = \sum_{j=1}^{n} L_{nj} A_n Q_j x$. Then $\|\tilde{A}_n - A_n\| \leq K \sum_{j=1}^{n} \epsilon_j$, so that $\|\tilde{A}_n\| \leq K$. Further

$\|B_n\tilde{A}_n - I_{X_n}\| \leq \frac{1}{2}$. Since $\tilde{A}_n(E_j) \subset Y_{nj}$ and $B_n(Y_{nj}) \subset E_j$ the operator $D_n = (B_n\tilde{A}_n)^{-1}$ leaves each E_j invariant. Let $\tilde{B}_n = D_nB_n$; then $\|\tilde{B}_n\| \leq 2C$.

The conclusion from these calculations, after relabelling, is that there is a constant C' so that for each n there is a Banach lattice Z_n with a band decomposition $Z_{n1} \oplus \cdots \oplus Z_{nn}$ such that $\dim Z_{nj} \leq d_j$ and operators $A_n \colon X_n \to Z_n$, $B_n \colon Z_n \to X_n$ so that $B_nA_n = I_{X_n}$, $\|A_n\| \leq 1$, $\|B_n\| \leq C'$ and $A_n(E_j) \subset Z_{nj}$, $B_n(Z_{nj}) \subset E_j$ for $1 \leq j \leq n$.

Let $p_n = \sum_{j=1}^n d_j$. We can alternatively regard Z_n as the space of sequences (ξ_j) so that $\xi_j = 0$ for $j > p_n$, with an associated norm. We can further suppose that the canonical basis vectors $(e_j)_{j=1}^{p_n}$ are normalized and that $Z_{nj} = [e_k]_{k=p_{j-1}+1}^{p_j}$.

Let $\mathcal{U}$ be a nonprincipal ultrafilter on the natural numbers $\mathbf{N}$. Define a norm $\|\ \|_Z$ on the space c_{00} of all finitely nonzero sequences by $\|\xi\|_Z = \lim_{\mathcal{U}} \|\xi\|_{Z_n}$. Let Z be the completion of c_{00} for this norm.

Let X_0 be the linear span of all (E_n) in X. We can define an operator $A \colon X_0 \to c_{00}$ by $Ax = \lim_{\mathcal{U}} A_nx$ and similarly $B \colon c_{00} \to X_0$ by $Bx = \lim_{\mathcal{U}} B_n\xi$. It is clear that $\|A\| \leq 1$ and $\|B\| \leq C'$. It is easy to verify that A isomorphically embeds X into Z in such a way that $A(E_n)$ is a complemented block UFDD. ∎

Remark: In [6] it is further claimed in Remark 2 that (under their additional hypotheses) if (E_n) is a *uniform* UFDD then we can choose Y so that (E_n) is a *uniform* complemented block UFDD. This of course would imply by Lemma 3.1 that one could find an unconditional basis for X by picking a basis of each E_n. However, no proof of Remark 2 is given and the natural proof does not appear to work. We shall see, however, that the claim of Remark 2 in [6] is nonetheless correct, but the proof is rather circuitous.

Let us now fix H as a complex N-dimensional Hilbert space, where $N \geq 2$. If $A \in \mathcal{L}(H)$ we denote its trace by tr A and its spectral radius by $r(A)$. We say that a subalgebra $\mathcal{A}$ of $\mathcal{L}(H)$ is **triangular** if every $A \in \mathcal{A}$ is of the form $A = \lambda I + S$ where S is nilpotent. This is equivalent to requiring that $r(A - \frac{1}{N}(\text{tr } A)I) = 0$ for all $A \in \mathcal{A}$. The following elementary lemma is very well-known and we include its proof only for reference.

LEMMA 3.4: *If $\mathcal{A}$ is a triangular subalgebra of $\mathcal{L}(H)$ then there is an orthonormal basis $(e_j)_{j=1}^N$ so that every $A \in \mathcal{A}$ with tr $A = 0$ is upper triangular, i.e. $(e_j, Ae_k) = 0$ whenever $j \leq k$.*

Proof: The subset $\mathcal{A}_0 = \{A \in \mathcal{A} : \text{tr } A = 0\} = \{A \in \mathcal{A}: \det A = 0\}$ is an ideal of $\mathcal{A}$. It suffices to construct an increasing sequence of subspaces $(E_k)_{0\le k\le N}$ with $\dim E_k = k$ so that $\mathcal{A}_0(E_k) \subset A_{k-1}$ for $1 \le k \le N$. Then we can construct an orthonormal basis $(e_k)_{k=1}^N$ with $e_k \in E_{N-k+1}$ for $1 \le k \le N$. Suppose then $E_0 = \{0\}$ to start the induction. Now suppose $1 \le k \le N$ and E_{k-1} has been constructed. Choose $x \notin E_{k-1}$ to minimize the dimension of $(\mathcal{A}_0 x + E_{k-1})/E_{k-1}$. If this dimension is zero then $\mathcal{A}_0 x \subset E_{k-1}$ and we let $E_k = [x, E_{k-1}]$. Otherwise there exists $S \in \mathcal{A}_0$ so that $Sx \notin E_{k-1}$. But then $\mathcal{A}_0 Sx + E_{k-1}$ contains Sx by minimality. Hence there exists $T \in \mathcal{A}_0$ with $Sx - TSx \in E_{k-1}$. However $(I - T)$ is invertible in $\mathcal{A}$ and E_{k-1} is $\mathcal{A}$-invariant so that $Sx \in E_{k-1}$ and we have a contradiction. ∎

Now suppose that $\mathcal{C}$ is a compact subset of $\mathcal{L}(H)$ which contains the identity $I = I_H$. Let $\|\mathcal{C}\| = \sup\{\|A\|: A \in \mathcal{C}\} \ge 1$. We define for $m \in \mathbf{N}$ the set $\mathcal{C}^{(m)}$ to be the set of all operators $T \in \mathcal{L}(H)$ of the form

$$T = \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \cdots \sum_{j_m=1}^{m} \alpha_{j_1,\dots,j_m} A_{j_1} \dots A_{j_m}$$

where $A_1, A_2, \dots, A_m \in \mathcal{C}$ and $|\alpha_{j_1,\dots,j_m}| \le 2^m$. Since $I \in \mathcal{C}$ the sets $\mathcal{C}^{(m)}$ are increasing compact sets and $\bigcup_m \mathcal{C}^{(m)}$ is the algebra generated by $\mathcal{C}$. If $T \in \mathcal{C}^{(m)}$ then $\|T\| \le (2m\|\mathcal{C}\|)^m$.

LEMMA 3.5: *Suppose $\delta > 0, M \ge 1, m, N \in \mathbf{N}$ with $N \ge 2$. Then there exists $p \in \mathbf{N}$ so that $p = p(M, N, m, \delta)$ has the following property: suppose $\mathcal{C}$ satisfies the above conditions with $\|\mathcal{C}\| \le M$. Suppose $S \in \mathcal{C}^{(m)}$ and $r(S - \frac{1}{N}(\text{tr } S)I) = \delta > 0$. Then there exists a projection P with $0 < \text{rank}\, P < N$ and $P \in \mathcal{C}^{(p)}$.*

Proof: Let $(\lambda_j)_{j=1}^N$ be the (complex) eigenvalues of S repeated according to algebraic multiplicity. We have $\max_{j,k} |\lambda_j - \lambda_k| \ge \delta$. It follows that we can reorder them so that for some s with $1 \le s \le N - 1$ we have $|\lambda_j - \lambda_k| \ge \delta/2N$ whenever $1 \le j \le s$ and $s + 1 \le k \le N$. Let $T = \prod_{j=1}^k (S - \lambda_j I)$. For each j we have $|\lambda_j| \le (2mM)^m$ and on multiplying out one has $T \in \mathcal{C}^{(q)}$ where q depends only on m, N and M. Let $\mu_k = \prod_{j=1}^s (\lambda_k - \lambda_j)$ so that $\mu_k = 0$ if $1 \le k \le s$ and $2^N(2mM)^{mN} \ge |\mu_k| \ge (\delta/(2N))^N$ if $s + 1 \le k \le N$. Next let $W = \prod_{k=s+1}^N (T - \mu_k I)$. Let $\gamma = \prod_{k=s+1}^N (-\mu_k)$. It is easily seen that $P = \gamma^{-1} W$ is a projection and that $0 < \text{rank}\, P < N$. From the obvious upper bound on

γ^{-1}, we obtain immediately that $P \in \mathcal{C}^{(p)}$ where p depends only on m, N, M, δ. ∎

The next estimate is crude and can doubtless be improved.

LEMMA 3.6: *Suppose H is a complex N-dimensional Hilbert space and that $\mathcal{A}$ is a triangular subalgebra of $\mathcal{L}(H)$. Let (A_k) be a sequence in $\mathcal{L}(H)$ with each A_k non-invertible such that $\sum_{k=1}^{\infty} A_k$ converges unconditionally to $I = I_H$. Let*

$$\sup_{|\alpha_k|\le 1} \|\sum_{k=1}^{\infty} \alpha_k A_k\| = M.$$

Then

$$\sup_{|\alpha_k|\le 1} d(\sum_{k=1}^{\infty} \alpha_k A_k, \mathcal{A}) \ge 2^{-3N^2}(N!)^{-N} M^{1-N^2}.$$

Proof: Let $b = \sup_{|\alpha_k|\le 1} d(\sum_{k=1}^{\infty} \alpha_k A_k, \mathcal{A})$. First observe that since $\mathcal{A}$ is triangular we can choose an orthonormal basis $(e_j)_{j=1}^{N}$ so that, when represented as matrices, each $B \in \mathcal{A}$ is of the form $B = \lambda I + S$ where S has an upper triangular matrix, i.e. $S = (s_{ij})$ where $s_{ij} = 0$ if $i \le j$.

Next, note that there exists $B \in \mathcal{A}$ so that $\|I - B\| \le b$. If $B = \lambda I + S$ then $\operatorname{tr} B = N\lambda$ and so $|\lambda - 1| \le b$. Then $\|I - S\| \le (b + \tau)$, where $\tau = |\lambda|$. Clearly $\|S\| \le 1 + b + \tau \le 4M$. Now expand $(S + (I - S))^N$; since $S^N = 0$ we obtain $1 \le 2^N(b+\tau)(4M)^{N-1} = 2^{3N-2}(b+\tau)M^{N-1}$.

We now estimate τ. Let $A_n = \{a_{ij}^n\}_{i,j=1}^{N}$. Then

$$\begin{aligned}\frac{1}{N}|\operatorname{tr} A_n| &\le \frac{1}{N}\sum_{i=1}^{N} |a_{ii}^n| \\ &\le (\prod_{i=1}^{N} |a_{ii}^n|)^{1/N} + \max_{i>j} |a_{ii}^n - a_{jj}^n| \\ &\le (\prod_{i=1}^{N} |a_{ii}^n|)^{1/N} + \sum_{i>j} |a_{ii}^n - a_{jj}^n|.\end{aligned}$$

Notice that for fixed i, j we have $\sum_{n=1}^{\infty} |a_{ii}^n - a_{jj}^n| \le 2b$. Hence on summing we have

$$\tau \le \sum_{n=1}^{\infty} |\prod_{i=1}^{N} a_{ii}^n|^{1/N} + N^2 b.$$

Now since det $A_n = 0$ we have that

$$\prod_{i=1}^{N} |a_{ii}^n| \leq \sum_{\sigma\in\Pi'} \prod_{i=1}^{N} |a_{i,\sigma(i)}^n|$$

where Π' is the collection of all permutations other than the identity of $\{1,2,\ldots,N\}$. Hence

$$\sum_{n=1}^{\infty}(\prod_{i=1}^{N} |a_{ii}^n|)^{1/N} \leq \sum_{\sigma\in\Pi'} \sum_{n=1}^{\infty}(\prod_{i=1}^{N} |a_{i,\sigma(i)}^n|)^{1/N}.$$

Let us fix $\sigma \in \Pi'$. Then

$$\sum_{n=1}^{\infty}(\prod_{i=1}^{N} |a_{i,\sigma(i)}^n|)^{1/N} \leq \prod_{i=1}^{N}(\sum_{n=1}^{\infty} |a_{i,\sigma(i)}^n|)^{1/N}.$$

Now if $i < \sigma(i)$ then

$$\sum_{n=1}^{\infty} |a_{i,\sigma(i)}^n| \leq b$$

so that since σ is not the identity, we obtain an upper estimate

$$\sum_{n=1}^{\infty}(\prod_{i=1}^{N} |a_{i,\sigma(i)}^n|)^{1/N} \leq M^{1-1/N}b^{1/N}.$$

Summing over all such permutations and combined with our previous estimates we finally obtain:

$$\tau \leq N^2 b + (N!)M^{1-1/N}b^{1/N}.$$

It follows that

$$b + \tau \leq 4(N!)M^{1-1/N}b^{1/N}$$

and the lemma follows. ∎

LEMMA 3.7: *Suppose $2 \leq N \in \mathbf{N}$ and $M \geq 1$. Then there is an integer $p = p(N,M)$ with the following property: suppose H is a complex N-dimensional Hilbert space and let (A_k) be a sequence in $\mathcal{L}(H)$ with each A_k non-invertible such that $\sum_{k=1}^{\infty} A_k$ converges unconditionally to $I = I_H$. Let $\mathcal{C}$ be the collection of all operators of the form $\sum_{k=1}^{\infty} \alpha_k A_k$ where $|\alpha_k| \leq 1$ and suppose $\|\mathcal{C}\| \leq M$. Then there is a projection P with $0 < \operatorname{rank} P < N$ and $P \in \mathcal{C}^{(p)}$.*

Proof: We argue by contradiction. Suppose for some M, N the result is false. Then we can find a sequence of such expansions $I = \sum_{k=1}^{\infty} A_{nk}$ so that the associated compact sets $\mathcal{C}_n = \{\sum_{k=1}^{\infty} \alpha_k A_{nk} \colon |\alpha_k| \le 1\}$ satisfy $\|\mathcal{C}_n\| \le M$ and that there is no nontrivial projection in $\mathcal{C}_n^{(n)}$. By passing to a subsequence we can further suppose that $\mathcal{C}_n$ converges in the Hausdorff metric to a compact set $\mathcal{C}$. It then follows from Lemma 3.5 that for each p we must have

$$\lim_{n\to\infty} \sup_{S\in\mathcal{C}_n^{(p)}} r\left(S - \frac{\operatorname{tr} S}{N} I\right) = 0.$$

Since all these quantities are continuous it follows that if $S \in \mathcal{C}^{(p)}$ for any p, we have

$$r\left(S - \frac{1}{N}(\operatorname{tr} S)I\right) = 0$$

and so the algebra $\mathcal{A}$ generated by $\mathcal{C}$ is triangular. But the preceding Lemma 3.6 now implies that

$$\inf_n \sup_{A\in\mathcal{C}_n} d(A, \mathcal{A}) > 0$$

which contradicts the fact that $\mathcal{C}_n$ converges in the Hausdorff metric to $\mathcal{C} \subset \mathcal{A}$. ∎

THEOREM 3.8: *Let X be a real or complex Banach space with local unconditional structure. Suppose that X has a uniform UFDD (E_n). Then there is an unconditional basis $(f_{nj})_{j=1}^{\dim E_n}$ of each E_n so that $(f_{nj})_{n,j}$ is an unconditional basis for X.*

Proof: We first prove the complex case. We shall prove the formally weaker statement that if X has l.u.st. and an N-UFDD (E_n) then there is a bounded projection Q on X so that $Q(E_n) \subset E_n$ for each n and $0 < \dim Q(E_n) < N$ for each N. Once this is proved the result follows simply by induction on $\sup_n \dim E_n$.

We first note that it is possible by Proposition 3.3 to regard (E_n) as a complemented block UFDD in a Banach space Y with unconditional basis (e_n). We suppose that $E_n \subset [e_k]_{k=r_{n-1}+1}^{r_n}$ where $r_0 < r_1 < \cdots$. Let $P\colon Y \to X$ be the associated projection. Let H be an N-dimensional Hilbert space and suppose for each n, $V_n\colon H \to E_n$ is an isomorphism satisfying $\|V_n\|\|V_n^{-1}\| \le \sqrt{N}$.

Letting (e_k^*) be the biorthogonal functions for the basis, we define for $r_{n-1}+1 \le k \le r_n$, $A_{nk}\colon H \to H$ by $A_{nk} = V_n^{-1}P(e_k^* \otimes e_k)V_n$. Let

$$\mathcal{C}_n = \left\{ \sum_{k=r_{n-1}+1}^{r_n} \alpha_k A_{nk} \colon |\alpha_k| \le 1 \right\}.$$

Then note that $\sup_n \|\mathcal{C}_n\| < \infty$ and so there is an integer p such that each $\mathcal{C}_n^{(p)}$ contains a projection R_n, with $0 < \operatorname{rank} R_n < N$.

Next observe that if $(B_n)_{n\in\mathbf{N}}$ is any sequence in $\mathcal{L}(H)$ with $B_n \in \mathcal{C}_n$ then the operator B defined on X by $Bx = V_n B_n V_n^{-1} x$ for $x \in E_n$ is bounded. It follows easily that the same statement is true if $B_n \in \mathcal{C}_n^{(q)}$ for fixed q. Hence the operator $Q\colon X \to X$ defined by $Qx = V_n R_n V_n^{-1}$ for $x \in E_n$ is bounded and the proof is complete in the complex case.

We now turn to the real case. Let $Q_n\colon X \to E_n$ be the natural projections. We complexify X to a space $\tilde{X}$, by norming $(x+iy)$ for $x, y \in X$ by

$$\|x+iy\|_c = \sup_{0\le\theta_n\le 2\pi} \|\sum_{k=1}^{\infty}(Q_k x \cos\theta_k + Q_k y \sin\theta_k)\|.$$

Now the subspaces $\tilde{E}_n = E_n + iE_n$ form a UFDD for $\tilde{X}$ and so we can pick an unconditional basis $(\phi_{nj})_{j=1}^{\dim E_n}$ in each E_n so that $(\phi_{nj})_{n,j}$ is an unconditional basis of $\tilde{X}$. Next let Y be the underlying real space for $\tilde{X} = X \oplus X$. Then Y has an unconditional basis $(\phi_{nj}, i\phi_{nj})_{n,j}$. Now the original (E_n) is a uniform complemented block UFDD in Y with this basis and so we complete the proof by applying Lemma 3.1. ∎

Let us give a sample application of this result. Let ω be the space of all sequences. Suppose X is a super-reflexive (Köthe) sequence space (so that the canonical basis vectors (e_n) form a 1-unconditional basis of X) and let $\Omega\colon X \to \omega$ be a (homogeneous) centralizer, i.e. a map satisfying, for a suitable constant Δ:

(1) $\Omega(\alpha x) = \alpha\Omega(x)$ for $\alpha \in \mathbf{R}$ and $x \in X$.

(2) $\|\Omega(ux) - u\Omega(x)\|_X \le \Delta\|u\|_\infty\|x\|_X$ for $x \in X$ and $u \in \ell_\infty$.

See [7] and [8] for discussion and examples. The simplest examples are those discussed in [10] of maps

$$\Omega(x)(n) = x(n) f\left(\frac{\log|x(n)|}{\|x\|_X}\right)$$

where $f\colon \mathbf{R} \to \mathbf{R}$ is a Lipschitz function (here we interpret the right-hand side as 0 if $x(n) = 0$). In the case $f(t) = t$ and $X = \ell_2$ one recovers the space Z_2 studied in [10] and [6].

We can now form the twisted sum $Y = X \oplus_\Omega X$ of all pairs (x, y) in $\omega \times \omega$ such that

$$\|(x,y)\|_\Omega = \|x\|_X + \|y - \Omega(x)\|_X < \infty.$$

This is a quasinorm, but is equivalent to a norm, since the space X is super-reflexive (as in [10]). The space Y is then a reflexive Banach space with a 2-UFDD (E_n) where E_n is the span of $(e_n, 0)$ and $(0, e_n)$. The vectors $(0, e_n)$ span a closed subspace X_0 of Y isomorphic to X and the quotient space Y/X_0 is also isomorphic to X. It may be shown that X_0 is complemented in Y, so that Y splits as a direct sum $X \oplus X$, if and only if there is a linear centralizer $L: X \to \omega$ (i.e. $Lx = bx$ for some $b \in \omega$) so that $\|Lx - \Omega x\|_X \leq C\|x\|_X$ for all $x \in X$. Such twisted sums arise very naturally as derivatives of complex interpolation scales of sequence spaces. If Z_0, Z_1 are two super-reflexive sequence spaces and $Z_\theta = [Z_0, Z_1]_\theta$ for $0 < \theta < 1$ is the usual interpolation space by the Calderón method, one can define a derivative dX_θ which is a twisted sum $X_\theta \oplus_\Omega X_\theta$ which splits if and only if $Z_1 = wZ_0$ for some weight sequence $w = (w(n))$ where $w(n) > 0$ for all n. These remarks follow easily from the methods of [7].

Our main conclusion here is that twisted sums of this type have l.u.st. if and only if they split as a direct sum. This extends the special case of Z_2 given in [6].

THEOREM 3.9: *Suppose X is a super-reflexive sequence space and $\Omega: X \to \omega$ is a centralizer on X. Let $Y = X \oplus_\Omega X$. Then the following are equivalent:*

(1) *Y is isomorphic to $X \oplus X$.*
(2) *Y has local unconditional structure.*
(3) *Y has an unconditional basis.*
(4) *The subspace X_0 is complemented in Y.*
(5) *There exists $b \in \omega$ and $C > 0$ so that $\|\Omega(x) - bx\|_X \leq C\|x\|_X$ for all $x \in X$.*

Proof: We have already observed the equivalence of (4) and (5). Clearly (4) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2). It remains only to show that (2) $\Rightarrow$ (4).

Let us first remark that we can assume that the canonical basis (e_n) of X is normalized; we can also assume that X is p-convex and q-concave with constant one, for suitable $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. We note that Y is super-reflexive and has a 2-UFDD (E_n) with the property that for suitable $x_n \in E_n$ the unconditional basic sequence (x_n) is equivalent to the canonical basis of the sequence space X and the induced unconditional basis (y_n) of the quotient $Y/[x_n]$ is also equivalent to the canonical basis of X.

If Y has l.u.st. then by Theorem 3.8 we can pick a normalized basis of E_n, say (f_n, g_n), so that $(f_n, g_n)_{n=1}^\infty$ is an unconditional basis of Y. We may suppose that $x_n = a_n f_n + b_n g_n$ where $|a_n| \geq b_n \geq 0$. If we consider the dual basis (f_n^*, g_n^*)

then the sequence $x_n^* = b_n f_n^* - a_n g_n^*$ must be equivalent to the canonical basis of the dual sequence space X^*.

It will now be convenient to switch to sequence space language. Let W be a sequence space so that if $\xi \in W$ then $\|\xi\|_W$ is equivalent to $\|\sum_{k=1}^\infty \xi(k) f_k\|$, and similarly let Z be a sequence space so that $\|\xi\|_Z$ is equivalent to $\|\sum_{k=1}^\infty \xi(k) g_k\|$. Since Y is super-reflexive we can assume that both W and Z are p-convex and q-concave with constant one (possibly changing the original choice of p, q), and that the canonical bases are normalized in both W and Z.

It is now easy to see that for a suitable constant C we have the inequalities

$$\frac{1}{C}\|\xi\|_X \leq \max(\|\xi\|_W, \|b\xi\|_Z) \leq C\|\xi\|_X$$

and

$$\frac{1}{C}\|\xi^*\|_{X^*} \leq \max(\|b\xi^*\|_{W^*}, \|\xi^*\|_{Z^*}) \leq C\|\xi^*\|_{X^*}$$

whenever $\xi, \xi^* \in c_{00}$. Note also that $b \in \ell_\infty$. We will show that these inequalities imply that W and Z both coincide up to equivalence of norm with X and hence that the basic sequences $(f_n), (g_n)$ and (x_n) are all equivalent. This suffices to show that $[x_n]$ is indeed complemented in Y, i.e. X_0 is complemented in Y.

Before proceeding we will need a lemma:

LEMMA 3.10: *Suppose V is a p-convex sequence space with $1 < p < \infty$, and that $0 \leq \xi, \xi^* \in c_{00}$ with $\langle \xi, \xi^* \rangle = \|\xi\|_V = \|\xi^*\|_{V^*} = 1$. Suppose further that $0 \leq \eta \in c_{00}$ with $\langle \eta, \xi^* \rangle \geq 1$ and $\|\eta\|_V = M$. Then, if $\frac{1}{p} + \frac{1}{q} = 1$, $\langle \min(\xi, \eta), \xi^* \rangle \geq q^{-1} M^{-q}$.*

Proof of Lemma: Note that if $t \geq 0$,

$$\langle \max(\xi, t\eta), \xi^* \rangle \leq (1 + M^p t^p)^{1/p}$$

and so

$$\langle \min(\xi, t\eta), \xi^* \rangle \geq 1 + t - (1 + M^p t^p)^{1/p}.$$

Now let $t = M^{-q} \leq 1$; the lemma follows by elementary estimates. ∎

Proof of the Theorem: (2) implies (4): We observe first that if $\xi \in c_{00}$ then $\|\xi\|_W \leq C\|\xi\|_X \leq C^2\|\xi\|_Z$.

Suppose first that M is chosen so large that $2qC^{4q+2} + M(1 - \frac{1}{2^q})^{1/q} < M$. Let $\beta > 0$ be chosen so that $\beta < \min(C^{-2}, (M+1)^{-1})$. We split $\mathbf{N}$ as $\mathcal{M}' \cup \mathcal{M}$ where

$\mathcal{M} = \{n: b_n \leq \beta\}$ and $\mathcal{M}' = \{n: b_n > \beta\}$. First suppose $\xi \in c_{00}$ is supported on $\mathcal{M}'$. Then

$$\max(\|\xi\|_W, \|\xi\|_Z) \leq C\beta^{-1}\|\xi\|_X$$

and if $\xi^* \in c_{00}(\mathcal{M}')$, then

$$\max(\|\xi^*\|_{W^*}, \|\xi^*\|_{Z^*}) \leq C\beta^{-1}\|\xi^*\|_{X^*}.$$

These inequalities show that on $c_{00}(\mathcal{M}')$ the spaces X, W, Z coincide.

Now let κ_n be the supremum of $\|\xi\|_Z$ subject to $\xi \in c_{00}(\mathcal{M})$, $\|\xi\|_W = 1$ and ξ has support of cardinality at most n. Then $\kappa_1 = 1$ and $\kappa_{n+1} \leq \kappa_n + 1$. We will show by induction that $\kappa_n \leq M$ for all n.

Suppose $\kappa_{n-1} \leq M$ and $\kappa_n > M$. Then there exists $\xi \geq 0$ with support of exactly n so that $\xi \in c_{00}(\mathcal{M})$, $\|\xi\|_W = 1$ and $\|\xi\|_Z > M$. Now $\|\xi\|_Z \leq M + 1$ so that $\|b\xi\|_Z \leq \beta(M+1) < 1$. Hence we must have $\|\xi\|_X \leq C$. Pick $0 \leq \xi^*$ with the same support so that $\langle \xi, \xi^* \rangle = 1$ and $\|\xi^*\|_{W^*} = 1$. Then $\|\xi^*\|_{X^*} \geq C^{-1}$, but $\|b\xi^*\|_{W^*} \leq \beta < C^{-2}$, so that $\|\xi^*\|_{Z^*} \geq C^{-2}$. It follows that we can pick $0 \leq \eta$ again with the same support, so that $\|\eta\|_Z \leq C^2$ and $\langle \eta, \xi^* \rangle = 1$. We then have $\|\eta\|_W \leq C^4$, and we can apply the lemma to see that

$$\langle \min(\xi, \eta), \xi^* \rangle \geq q^{-1}C^{-4q}.$$

Now let $\zeta(n) = \xi(n)$ whenever $\xi(n) \leq 2qC^{4q}\eta(n)$ and $\zeta(n) = 0$ otherwise. It follows easily that $\langle \zeta, \xi^* \rangle \geq \frac{1}{2}$ (so that $\|\zeta\|_W \geq \frac{1}{2}$) and $\|\zeta\|_Z \leq 2qC^{4q}\|\eta\|_X \leq 2qC^{4q+2}$.

Now

$$\begin{aligned}\|\xi\|_Z &\leq \|\zeta\|_Z + \kappa_{n-1}\|\xi - \zeta\|_W \\ &\leq 2qC^{4q+2} + M(1 - \|\zeta\|_W^q)^{1/q} \\ &\leq 2qC^{4q+2} + M(1 - \frac{1}{2^q})^{1/q} \\ &\leq M.\end{aligned}$$

This contradiction yields the result that $\kappa_n \leq M$ for all n and hence the theorem. ∎

Remark: The most natural case of Theorem 3.9 is when $X = \ell_2$ so that Y is a "twisted Hilbert space", i.e. Y has a Hilbertian subspace X_0 so that Y/X_0 is Hilbertian. The result suggests the conjecture that every twisted Hilbert space with an unconditional basis is a Hilbert space.

4. An example

In this final section we construct an explicit example of a non-Hilbertian Orlicz sequence space where every closed subspace with a uniform UFDD has local unconditional structure. In [13] Komorowski and Tomczak-Jaegermann show that if a Banach space with an unconditional basis is not hereditarily Hilbertian then it has a closed subspace failing l.u.st. but with a uniform UFDD.

We define a function G on $[0,\infty)$ by $G(0)=0$ and:

$$G(x)=\begin{cases} x(1-\frac{1}{2}\log x), & 0<x\leq 1,\\ \sqrt{x}, & x>1.\end{cases}$$

Note that G is differentiable on $(0,\infty)$ and

$$G'(x)=\begin{cases} \frac{1}{2}(1-\log x), & 0<x\leq 1,\\ \frac{1}{2\sqrt{x}}, & x>1.\end{cases}$$

LEMMA 4.1: *For any $1\leq p<\infty$, whenever $(a_n)_{n=0}^{\infty}$ is a sequence with $0\leq a_n\leq 1$ and $(\sum_{n=1}^{\infty} a_n^p)^{1/p}\leq 1$ then for any sequence (t_n) with $t_n\geq 0$ we have:*

$$\sum_{n=1}^{\infty} G(a_nt_n)\leq G(\sum_{n=1}^{\infty} a_nt_n)+p\sum_{n=1}^{\infty} t_nG(a_n).$$

Proof: It clearly suffices to prove the inequality for a finite sequence $(a_1,\dots,a_N)$ of strictly positive numbers.

Suppose $\lambda>p$. Consider the function

$$\Phi(t_1,\dots,t_n)=\sum_{n=1}^{N} G(a_nt_n)-G(\sum_{n=1}^{N} a_nt_n)-\lambda\sum_{n=1}^{N} t_nG(a_n),$$

defined on the positive cone $\{t\colon t_i\geq 0,\ 1\leq i\leq N\}$. Note first that if $a_nt_n\geq 1$ then $G(a_nt_n)=a_n^{1/2}t_n^{1/2}\leq a_nt_n\leq t_nG(a_n)$. It follows quickly that Φ is bounded above by its maximum on the set $\{t\colon a_it_i\leq 1,\ 1\leq i\leq N\}$.

Let Φ attain its maximum at the point s where $0\leq s_i\leq a_i^{-1}$ for $1\leq i\leq N$. Let $S=\sum_{j=1}^{N} a_js_j$. For any index j such that $s_j>0$ we have

$$a_jG'(a_js_j)-a_jG'(S)-\lambda G(a_j)=0.$$

Since $a_js_j\leq 1$ this simplifies to

$$\frac{1}{2}(1-\log a_js_j)-G'(S)=\lambda(1-\frac{1}{2}\log a_j)$$

and hence to

$$\log a_j s_j + 2G'(S) = 1 - 2\lambda + \lambda \log a_j.$$

Assume that the set J of indices such that $s_j > 0$ is nonempty. Then taking exponentials and summing

$$e^{2G'(S)}S = e^{1-2\lambda} \sum_{j\in J} a_j^{\lambda}.$$

Now if $S \geq 1$ then $e^{2G'(S)}S \geq S$. If $S < 1$ then $e^{2G'(S)}S = e$. In either case we deduce that

$$\sum_{j\in J} a_j^{\lambda} \geq e^{2\lambda-1} > 1,$$

and this contradicts the conditions on $(a_1, \ldots, a_n)$. Now since J is empty the maximum is attained at the origin and is 0. Since $\lambda > p$ is arbitrary the lemma is proved. ∎

Now let F be the Orlicz function defined by $F(0) = 0$ and

$$F(x) = \begin{cases} x^2(1 - \log x), & 0 < x \leq 1, \\ x, & x > 1. \end{cases}$$

The function F is convex for $x \leq e^{-1/2}$ so that F is equivalent at 0 to a convex function. We will consider the Orlicz sequence space ℓ_F. The norm defined in the usual way

$$\|x\|_F = \inf\{t > 0\colon \sum_{n=1}^{\infty} F(x(n)/t) \leq 1\}$$

is, strictly speaking, only a quasi-norm but is equivalent to a norm. Note that ℓ_F is reflexive and has cotype 2 and type p for any $p < 2$; these facts are easily computed from the function F. Clearly $\ell_F \subset \ell_2$ and $\|x\|_2 \leq \|x\|_F$ for all $x \in \ell_F$. We will also consider the modular Λ defined on ℓ_F by

$$\Lambda(x) = \sum_{n=1}^{\infty} F(|x(n)|).$$

It will be convenient to introduce for $0 \leq a \leq 1$ the Orlicz functions F_a where $F_a(0) = 0$ and

$$F_a(x) = \begin{cases} x^2(1 - a\log x), & 0 < x \leq 1, \\ x, & x > 1. \end{cases}$$

If (a_n) is any sequence with $0 \leq a_n \leq 1$ we will consider the Orlicz modular space (or Orlicz–Musielak space) $\ell(F_{a_n})$ of all sequences $x(n)$ so that $\sum_{n=1}^{\infty} F_{a_n}(|x(n)|) < \infty$ again with the (quasi-)norm defined in the usual way.

LEMMA 4.2: *If $M \geq 1$ there is a constant $K = K(M)$ with the following property. Whenever $(x_n)_{n\in\mathcal{M}}$ and $(y_n)_{n\in\mathcal{M}}$ are two (finite or infinite) M-unconditional basic sequences satisfying the conditions:*

$$\sup_{n\in\mathcal{M}} \max\left(\frac{\|x_n\|_F}{\|y_n\|_F}, \frac{\|y_n\|_F}{\|x_n\|_F}\right) \leq M,$$

$$\sup_{n\in\mathcal{M}} \max\left(\frac{\|x_n\|_2}{\|y_n\|_2}, \frac{\|y_n\|_2}{\|x_n\|_2}\right) \leq M,$$

then $(x_n)_{n\in\mathcal{M}}$ and $(y_n)_{n\in\mathcal{M}}$ are K-equivalent.

Proof: We first note that it will suffice to consider normalized bases. Suppose that (x_n) is a normalized block basic sequence. Let $a_n = \|x_n\|_2^2$. Then for any finitely nonzero $(t_n)_{n\in\mathcal{M}}$ with $\max|t_n| \leq 1$, we have:

$$\begin{aligned}\Lambda(\sum_{n\in J} t_n x_n) &= \sum_{n\in\mathcal{M}} \sum_{j=1}^{\infty} F(|t_n||x_n(j)|) \\ &= \sum_{n\in\mathcal{M}} \sum_{j=1}^{\infty} (|t_n|^2 F(|x_n(j)|) - |x_n(j)|^2 |t_n|^2 \log|t_n|) \\ &= \sum_{n\in\mathcal{M}} |t_n|^2 (1 - a_n \log|t_n|).\end{aligned}$$

It follows easily that (x_n) is 1-equivalent to the unit vector basis in the Orlicz modular space $\ell(F_{a_n})$.

Now it is clear that if $a \leq b \leq Ma$ then $F_a(x) \leq F_b(x) \leq MF_a(x)$ and from this it follows easily, using the uniform Δ_2-conditions on F_t for $0 \leq t \leq 1$, that there is constant $K = K(M)$ so that if $a_n \leq b_n \leq Ma_n$ for $n \in \mathcal{M}$ then the unit vector bases of $\ell(F_{a_n})$ and $\ell(F_{b_n})$ are K-equivalent.

Now we turn to the general case. First note that ℓ_F is super-reflexive and so if X is any closed subspace and $2 < p < \infty$ is fixed then X^* is of cotype p with some cotype constant D independent of X. Now suppose $(x_n)_{n\in\mathcal{M}}$ is a normalized M-unconditional basic sequence in ℓ_F whose closed linear span is a subspace X. Consider the co-ordinate functional e_j^* as an element of X^*. Then

$$\sum_{n\in\mathcal{M}} |x_n(j)|^p = \sum_{n\in\mathcal{M}} |e_j^*(x_n)|^p \leq D^p M^p.$$

Suppose $(t_n)_{n\in\mathcal{M}}$ is finitely nonzero, and $\max|t_n| \leq 1$. Since ℓ_F has cotype 2, there is (cf. [16] Theorem 1.d.6) a universal constant C so that

$$\frac{1}{CM}\|(\sum_{n\in\mathcal{M}} |t_n|^2 |x_n|^2)^{1/2}\|_F \leq \|\sum_{n\in\mathcal{M}} t_n x_n\|_F \leq CM\|(\sum_{n\in\mathcal{M}} |t_n|^2|x_n|^2)^{1/2}\|_F.$$

Let us calculate the modular $\Lambda(f)$ where $f = (\sum_{n\in\mathcal{M}} |t_n|^2|x_n|^2)^{1/2}$. Then

$$\Lambda(f) = \sum_{j=1}^{\infty} G(\sum_{n\in\mathcal{M}} |t_n|^2|x_n(j)|^2).$$

Since G is concave, it is subadditive and so:

$$\Lambda(f) \leq \sum_{j=1}^{\infty}\sum_{n\in\mathcal{M}} G(|t_n|^2|x_n(j)|^2) = \sum_{n\in\mathcal{M}} F_{a_n}(|t_n|).$$

For the reverse inequality consider

$$\Lambda(M^{-1}D^{-1}f) = \sum_{j=1}^{\infty} G(\sum_{n\in\mathcal{M}} |t_n|^2 w_n(j))$$

where $w_n(j) = M^{-2}D^{-2}|x_n(j)|^2$. Then $(\sum_{n\in\mathcal{M}} w_n(j)^{p/2})^{2/p} \leq 1$. Thus we can apply Lemma 4.1 to deduce that for each j,

$$\sum_{n\in\mathcal{M}} G(|t_n|^2 w_n(j)) \leq G(\sum_{n\in\mathcal{M}} |t_n|^2 w_n(j)) + \frac{p}{2}\sum_{n\in\mathcal{M}} G(w_n(j))|t_n|^2.$$

Summing over j, and using the fact that $MD \geq 1$, we obtain

$$\sum_{n\in\mathcal{M}} F_{a_n}(M^{-1}D^{-1}|t_n|) \leq \Lambda(f) + \frac{p}{2}\sum_{n\in\mathcal{M}} |t_n|^2.$$

The fact that ℓ_F has cotype 2 implies an estimate

$$(\sum_{n\in\mathcal{M}} |t_n|^2)^{1/2} \leq CM\|\sum_{n\in\mathcal{M}} t_n x_n\|_F.$$

It follows easily that $(x_n)_{n\in\mathcal{M}}$ is K-equivalent to the unit vector basis of $\ell(F_{a_n})$ where K depends only on M. This and the preceding remarks complete the proof. ∎

The following theorem follows immediately:

THEOREM 4.3: *Every unconditional basic sequence in ℓ_F is equivalent to a sequence of constant coefficient blocks in ℓ_F and hence spans a subspace isomorphic to a complemented subspace of ℓ_F.*

Let us note that this implies a strong universality principle for unconditional basic sequences in ℓ_F. Precisely, ℓ_F has an unconditional basis (obtained by repeating every length constant coefficient block infinitely often) so that every normalized unconditional basic sequence is equivalent to a subsequence of the basis. Such a property is also enjoyed by Pełczyński's universal space ([15], Theorem 2.d.10 or [19]). We next observe that ℓ_F obeys a strong form of the Schroeder–Bernstein property for spaces with unconditional bases.

THEOREM 4.4: *Let X be a Banach space with an unconditional basis, and suppose that X embeds into ℓ_F and ℓ_F embeds into X. Then X is isomorphic to ℓ_F.*

Proof: By the preceding theorem X is isomorphic to a complemented subspace of ℓ_F spanned by constant coefficient blocks $(u_n)_{n=1}^{\infty}$. We now observe that $(u_n)_{n=1}^{\infty}$ must contain an infinite number of blocks of the same length, for otherwise X is isomorphic to an Orlicz modular space $\ell_{F_{a_n}}$ where $\lim_{n\to\infty} a_n = 0$ and this can easily be seen not to contain a copy of ℓ_F. Hence ℓ_F is complemented in X. By Proposition 3.a.5 of [15], ℓ_F is isomorphic to $\ell_F \oplus X$ and this is now trivially isomorphic to X. ∎

In [13] it is shown that any non-hereditarily Hilbertian space with an unconditional basis contains a closed subspace with a 2-UFDD which fails to have local unconditional structure. The following theorem (our main result of the section) shows that this result cannot be substantially improved.

THEOREM 4.5: *Let X be a closed subspace of ℓ_F with a UFDD $(E_k)_{k=1}^{\infty}$ such that the spaces (E_k) are uniformly Hilbertian (i.e. $\sup d(E_k, \ell_2^{N_k}) < \infty$, where $N_k = \dim E_k$.) Then one can choose an unconditional basis $(f_{ik})_{i=1}^{N_k}$ of E_k so that the collection $(f_{ik})_{i,k}$ is an unconditional basis of X.*

Remark: In particular the theorem applies to any uniform-UFDD.

Proof: Let $\| \, \|_{E_k}$ be a Euclidean norm on E_k so that $\|x\|_F \le \|x\|_{E_k} \le C\|x\|_F$ where $C = \sup d(E_k, \ell_2^{N_k})$. Let M be the constant of unconditionality for the Schauder decomposition (E_k). We choose a basis (f_{ik}) for E_k which is orthonormal for both $\| \, \|_{E_k}$ and $\| \, \|_2$. Suppose (t_{ik}) is finitely non zero and that (ϵ_{ik}) is a choice of signs. Let $x_k = \sum_{i=1}^{N_k} t_{ik} f_{ik}$ and $y_k = \sum_{i=1}^{N_k} \epsilon_{ik} t_{ik} f_{ik}$. Then for the set $\mathcal{M}$ of all k such that they are nonzero we have $\|x_k\|_F / \|y_k\|_F \le C$ and $\|y_k\|_F / \|x_k\|_F \le C$. We also have $\|x_k\|_2 = \|y_k\|_2$. Both $(x_k)_{k\in\mathcal{M}}$ and $(y_k)_{k\in\mathcal{M}}$ are M-unconditional basic sequences. Hence they are K-equivalent by Lemma

4.2 where $K = \dot{K}(C, M)$. In particular,

$$\| \sum_{k \in \mathcal{M}} y_k \|_F \leq K \| \sum_{k \in \mathcal{M}} x_k \|_F$$

whence the basis (f_{ik}) is K-unconditional. ∎

Remark: The properties of unconditional basic sequences in ℓ_F have other applications, for example to uniqueness questions. We plan to discuss these applications in a separate paper.

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